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CONCEPTUAL KNOWLEDGE FALLS THROUGH THE CRACKS: COMPLEXITIES OF LEARNING TO TEACH MATHEMATICS FOR UNDERSTANDING

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In this article we focus on two interrelated aspects of the process of learning to teach mathematics for understanding: (a) ideas and practices for teaching procedural knowledge and (b) ideas and practices for teaching conceptual knowledge. We explore one student teacher's ideas and practices, together with the messages about teaching for procedural and conceptual knowledge that were presented by the teacher education program in which the student teacher was enrolled and the placement schools in which she student taught. We reveal a pattern in which the student teacher, her mathematics methods course instructor, her cooperating teachers, and the administrators of her placement schools expressed a variety of strong commitments to teaching for both procedural and conceptual knowledge; but with these commitments, the student teacher taught, learned to teach, and had opportunities to learn to teach for procedural knowledge more often and more consistently than she did for conceptual knowledge. We find that the actual teaching pattern (what was done) was the product of unresolved tensions within the student teacher, the other key actors in her environment, and the learning-to-teach environment itself. We hypothesize that situational supports constructed to emphasize more consistently teaching for conceptual knowledge might help resolve at least some of the tensions, and we suggest that such supports should be developed if the national goal to increase the teaching of mathematics for understanding is to be achieved.

Teaching mathematics for understanding is one of the hallmarks of current reform efforts in mathematics teacher education. Numerous commissions (Cockcroft, 1982; Collins, 1988; Howson & Wilson, 1986; Mathematical Sciences Education Board, 1991) and professional organizations (e.g., Mathematical Association of America, 1991; National Council of Teachers of

This article extends the analysis of learning to teach mathematics that we began in Borko et al. (1992), "Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily?" The previous article focused on the experiences of one student teacher as she struggled to teach division of fractions. This article sets that struggle in a larger context of learning to teach for procedural and conceptual knowledge.

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Mathematics [NCTM], 1989) have called for teachers to devote more time and attention to developing students' understanding of mathematics. But teaching mathematics for understanding is an extremely complex process (Hiebert, 1986), and the mathematical and pedagogical skills and knowledge needed are considerable (Ball, 1991; McDiarmid, Ball, & Anderson, 1989).

In this article, we focus on two interrelated aspects of the process: (a) ideas and practices for teaching procedural knowledge and (b) ideas and practices for teaching conceptual knowledge (including the connections between procedural and conceptual knowledge) as they were worked out for and by novice middle school mathematics teachers in our study, "Learning to Teach Mathematics." We explore the patterns in one student teacher's ideas and practices related to teaching for procedural and conceptual knowledge. We then explore messages about teaching for procedural and conceptual knowledge that were presented by the teacher education program in which the student teacher, "Ms. Daniels" (all names used for study participants are pseudonyms), was enrolled and the placement schools in which she student taught. The juxtaposition reveals how the patterns serve to create a context in which learning to teach for conceptual knowledge becomes quite problematic.

Procedural and Conceptual Knowledge

We use the terms procedural knowledge and conceptual knowledge to denote a distinction often made between two forms of mathematical knowing (Hiebert & Lefevre, 1986; Hiebert & Wearne, 1988). As commonly used, procedural knowledge refers to mastery of computational skills and knowledge of procedures for identifying mathematical components, algorithms, and definitions (knowing how to identify a problem, in its broadest and most routine sense, and how to solve it correctly). More specifically, procedural knowledge of mathematics has two parts: (a) knowledge of the format and syntax of the symbol representation system and (b) knowledge of rules and algorithms, some of which are symbolic, that can be used to complete mathematical tasks. For example, a student with procedural knowledge of division of fractions will know how to write the problem on paper and the steps involved in the algorithm for completing the division (first invert the divisor, then multiply the new fractions). This procedure has other procedures embedded in it, for example, in what order to write the fractions, how to invert, and how to multiply fractions. Teaching division of fractions for procedural knowledge is exemplified by step-by-step presentation of rules and algorithms as well as strategies for remembering them.

Conceptual knowledge refers to knowledge of the underlying structure of mathematics—the relationships and interconnections of ideas that explain and give meaning to mathematical procedures. In the case of division of fractions, conceptual knowledge includes such ideas as the nature of fractions in general and of the particular fractions to be divided, as well as what it means to divide. Teaching division of fractions for conceptual knowledge is exemplified in (a) the use of concrete and semi-concrete models (e.g., Cuisenaire rods, egg carton pieces, circular or rectangular drawings) that illustrate or represent division of fractions and (b) discussion of the links between and among mathematical ideas (e.g., how measurement and partitive meanings of division, first explored with whole numbers, can be extended to fractions; how division of fractions is related to proportions or scaling; how multiplication is related to division; and how story problems are related to number sentences). Conceptual teaching of a topic such as division of fractions is intended to help students understand the mathematical procedures used to obtain correct answers.

Both procedural and conceptual knowledge are considered necessary aspects of mathematical understanding. Thus, to teach for mathematical understanding must include teaching for both procedural and conceptual knowledge (Wearne & Hiebert, 1988b).

However, there is evidence that procedural knowledge, specifically rote knowledge of rules and algorithms, is emphasized in most schools and that teachers devote much less time and attention to conceptual knowledge (Porter, 1989). Indeed, data from the fifth National Assessment of Educational Progress (Mullis, Dossey, Owen, & Phillips, 1991) and the earlier Second International Mathematics Study (McKnight et al., 1987) indicate that, at least in the United States, rotely learned rules and procedures dominate school mathematics. About this, Wearne and Hiebert (1988a) state, "The student who tries to make sense of [procedural] manipulations is something of an anomaly" (p. 220).

The apparent tendency to emphasize rote knowledge of mathematical procedures in U.S. schools, coupled with the national goal to improve students' understanding of mathematics, has led mathematics education reformers to stress the importance of preparing teachers to teach more consistently for conceptual knowledge. In our study, teaching for conceptual knowledge was a major theme of the mathematics education course work, an expressed commitment of the placement schools, and a part of the student teachers' own beliefs about good teaching. However, although the teachers and teacher educators we observed struggled to teach for conceptual knowledge, they often appeared to emphasize procedural knowledge instead. The purpose of this article is to reveal some of the tensions and pressures that face novice teachers and teacher educators who attempt to teach for conceptual knowledge and that make it difficult for them to move beyond procedural knowledge to conceptual knowledge in their classrooms.

In the sections that follow, we briefly describe the study of novice mathematics teachers that we conducted; then we turn to our data. To present the data, we begin with a mini–case study of Ms. Daniels, one of the student teachers in our study, to illustrate some of the tensions in Ms. Daniels's teaching practices and beliefs about mathematics and mathematics teaching as she tried, but generally failed, to teach for conceptual knowledge during the course of her student teaching year. After presenting the mini–case

study, we examine the tensions surrounding efforts to teach for conceptual and procedural knowledge that were evidenced in the university teacher education experiences and the student teaching placement schools in which Ms. Daniels participated. We find that the expectations and demands emanating from the university and the placement schools, regarding what should be taught and learned in classrooms, play a major role in determining how novices learn to teach for conceptual and procedural knowledge.

THE LEARNING TO TEACH MATHEMATICS STUDY

Our study, "Learning to Teach Mathematics," was designed to examine the process of becoming a middle school mathematics teacher by following a small number of novice teachers through their final year of teacher preparation and their first year of teaching. Our primary goal was to describe and understand the novice teachers' knowledge, beliefs, thinking, and actions related to the teaching of mathematics over the two-year period. Additional goals were to describe and explain the contexts for learning to teach created by the novice teachers' university teacher education experiences and their experiences in the public schools where they student taught (Year 1) and held their first teaching jobs (Year 2). Because the university and the public schools are the two major contexts for learning the culture and social organization of teaching in most teacher education programs, we assumed that these contexts would be the major sources of external influence on the process of learning to teach.

METHODS

The overall conceptual framework, data collection procedures, and data analysis strategies of the Learning to Teach Mathematics study have been recently described elsewhere (Borko et al., 1992). In this section, we briefly review the sources of data and analysis procedures used to obtain the material presented in this article.

Participants and Setting

Eight seniors in a K–8 teacher education program at a large southern university participated in the first year of the project. All 8 were members of a cohort of 38 students in a yearlong senior year experience that included professional course work and student teaching. The program was specifically intended for preservice teachers being certified for grades K–8 and who were primarily interested in middle school teaching. All 8 participants had selected mathematics as an area of concentration (consisting of approximately 20 semester hours of course work in mathematics, statistics, and computer science) and indicated an intention to teach middle school mathematics after graduation. Ms. Daniels, the participant whose teaching is analyzed in this article, had the most extensive mathematics background of any of the student teachers in the program, having completed her first 3 years at the university as a mathematics major (for more detail about Ms. Daniels's preparation in mathematics, see Borko et al., 1992).

The design of the teacher education program called for each cohort member to have four different student teaching placements (7 weeks each; two each semester) in a city unified school district of approximately 15 000 students. During the first three placements, the cohort spent 3–3 1/2 hours at a school each morning and took afternoon courses taught by university faculty; during the final placement, they taught the full school day. During the first 12 weeks of the academic year, mathematics, language arts, and reading methods courses were taught; during the second 12 weeks, courses in science and social studies methods and diagnosis were taught.

This article focuses on Ms. Daniels's teaching during her first, third, and fourth student teaching placements. Her first placement was in a self-contained sixth-grade classroom in an elementary school. We observed Ms. Daniels teaching mathematics on three occasions during that placement. The first two were tutorial lessons with an individual student. The primary activity on the third occasion was a game that Ms. Daniels created as a follow-up activity to a lesson on exponents taught by her cooperating teacher. Ms. Daniels's third assignment was with a mathematics teacher in a junior high school. Our four observations of her teaching took place in a seventh-grade honors mathematics class during a unit on number theory. Class sessions focused on topics such as prime and composite numbers, prime factorization, and factor trees. For her fourth placement, Ms. Daniels returned to the sixth grade, but to another classroom in a different elementary school. During the week of observation she taught mathematics twice a day: "Morning Math" from approximately 8:30 to 9:00 a.m. and the regular mathematics session from 11:30 a.m. to 12:30 p.m. We observed lessons on topics in geometry including perimeter, circumference, area, surface area, and volume during regular mathematics instruction. We also observed two Morning Math lessons, one covering division of fractions and the other conversion between ratios (fractions) and decimals.

Data Collection

Ms. Daniels's beliefs and knowledge. The primary source of information about Ms. Daniels's beliefs and knowledge was a baseline interview[1] administered at the beginning, middle, and end of the school year. Open-ended questions, many of which were based on vignettes describing hypothetical classroom situations involving mathematics, were intended to elicit her beliefs about, and knowledge of, mathematics, pedagogy, mathematics pedagogy, learning to teach, and other domains of teachers' professional activity (e.g., Shulman & Grossman, 1988).

Ms. Daniels's teaching (thinking and actions in the classroom). To gather information about Ms. Daniels's classroom teaching, we conducted week-long visits to her class near the end of her first, third, and fourth student teaching placements. Primary data sources were classroom observations and interviews before and after the observations. Classroom observations focused on Ms. Daniels's mathematics instruction (e.g., her explanations, demonstrations, and assignment of student tasks). In interviews before each observation, we asked her about her planning activities and the factors she considered when preparing the lesson (e.g., content, student characteristics, instructional materials, and her cooperating teacher's suggestions). Post-observation interviews asked for her reactions to the lesson and specific lesson components (e.g., selected explanations, demonstrations, examples, and student activities). These data were supplemented by copies of written lesson plans, worksheets, and other handouts.

University experience. To gather information about the university experience, we regularly observed the mathematics methods course, interviewed the instructor before and after each class session about his goals and objectives for the session and his reactions to it, and interviewed the student teachers about their reactions to the course. We also interviewed the student teachers, their methods instructors, the university supervisors, and the teacher education program director about their overall impressions of the university's teacher education program. To supplement these data, we collected documents pertaining to the teacher education program and to the students' progress in it.

Public school experience. The primary data source for this component was a set of interviews conducted with two district-level administrators (the associate superintendent for instruction and the mathematics supervisor), the principals, and the cooperating teachers with whom Ms. Daniels worked during her first, third, and fourth placements. Questions were designed to elicit the respondents' views of (a) decision processes regarding instructional materials, course content, assessment, and resources; and (b) expectations for teacher performance, working relationships, rewards, and sanctions, at the district, school, and classroom levels. Additional information was provided by observations of the cooperating teachers' mathematics instruction and interviews with Ms. Daniels about her cooperating teachers.

Data Analysis

These data were initially reviewed and coded according to a scheme we developed for the entire corpus of data from our study (see Borko et al., 1992, for details about our initial review and coding of the data). Based on

[1]The baseline interview is a modification of an interview developed by the National Center for Research on Teacher Education (NCRTE) at Michigan State University and was used with the Center's permission. See Ball and McDiarmid (1990) for information about the original interview protocol and NCRTE (1988) for a description of the research program for which it was developed.

an early reading of some of the data suggesting that there were patterns related to teaching for procedural and conceptual knowledge, we reviewed the initial coded categories for evidence of practices and ideas pertaining to teaching for procedural and conceptual knowledge. Following Spradley's (1980) model for thematic analysis, we identified a pattern of tensions and competing pressures in the data. Results of these analyses are presented in the following sections of the article.

MS. DANIELS'S BELIEFS AND KNOWLEDGE ABOUT *TEACHING FOR PROCEDURAL AND CONCEPTUAL KNOWLEDGE*

Analysis of the baseline interviews suggests that Ms. Daniels recognized the difference between procedural and conceptual knowledge and believed that both were necessary for understanding mathematics. However, Ms. Daniels could articulate her ideas about how to teach for procedural knowledge much better than she could her ideas about how to teach for conceptual knowledge, and her attempts to provide conceptual explanations for the hypothetical problems posed to her in the interview were weak. Further, she considered her own procedural knowledge of mathematics and mathematics teaching to be much stronger than her conceptual knowledge.

Ms. Daniels's Views of Procedural and Conceptual Knowledge

Ms. Daniels's beliefs about procedural and conceptual knowledge of mathematics were revealed by her responses to a series of questions about what it means for a student to be excellent in mathematics. In these responses Ms. Daniels made a distinction between arithmetic and mathematics that is similar to the literature's distinction between procedural and conceptual knowledge. She said,

> [Arithmetic is] numbers and using them in certain situations. I guess that's your basic skills like addition and subtraction, multiplication and division, ordering sequences and things like that that just involve numbers and their variables, no theory, theorems, or anything like that....I think mathematics is a combination of the arithmetic and how you apply the arithmetic along with the reasoning behind its application....[Knowing mathematics is] that deep understanding of the underlying meaning of all those things, the formulas.

Thus arithmetic, like procedural knowledge, meant knowing how to do algorithms and knowing how to apply them in constrained situations. Knowing mathematics, like conceptual knowledge, meant knowing concepts, theories, and the reasoning behind algorithms.

Ms. Daniels believed that doing arithmetic involved primarily memorized procedures. Doing mathematics seemed to require more than memorization, but Ms. Daniels was not explicit about what it involved. She attempted to articulate the differences between doing arithmetic and doing mathematics as follows.

> Math encompasses both the conceptual understanding as well as the arithmetic. Whereas to do arithmetic, you don't have to understand the concepts necessarily because a lot of arithmetic is done by memorization. Whereas conceptual understanding, that's using your brain—your thinking skills—at a much higher level.

As the quote above suggests, Ms. Daniels did not appear to be able to give precise descriptions of her conceptions of arithmetic and mathematics. One reason for this limitation may be that these conceptions, and the relationship between them, were not well developed in her thinking. She often seemed to be working out the details of the distinctions between the two types of knowledge as she talked about them during interviews.

However, despite her vagueness, Ms. Daniels steadfastly maintained that it was knowledge of arithmetic and mathematics together that constituted a real understanding of mathematics. She believed that with this understanding students could solve real-world problems and see the connections between and among topics in the curriculum.

Ms. Daniels's Beliefs About Teaching for Conceptual and Procedural Knowledge

The baseline interview data reveal that Ms. Daniels thought that teaching for procedural knowledge (arithmetic) and teaching for conceptual knowledge (mathematics) required different activities. She explained,

> Arithmetic is more guided. Your thinking, your thought process is guided. You're told to do it in a certain way and that's how you do it. Whereas mathematics, you create a lot of it on your own. It's a lot of your own thinking. It's not someone else's.

However, when asked to talk about how she would teach for each, she was clearer about teaching for procedural knowledge. When teaching arithmetic, according to Ms. Daniels, teachers must help students develop skills in accepted mathematical procedures. She expected teachers would do this by carefully demonstrating algorithms, explaining each step in detail, and then providing opportunities for students to practice the algorithms until they were "engraved in their brains." She found support for these teaching activities in the textbooks to which she was asked to react in the interviews. She approved of the textbooks' presentation of example problems to demonstrate algorithms and their carefully stated definitions. She praised the practice exercises provided for students, although she felt that there were often too few for students to become highly skilled, and that she would have to provide additional exercises from other sources.

Her ideas for teaching knowledge of the concepts, reasoning, and theories that constituted mathematics were vague. She thought that some students "can discover things on their own, relate things in their own mind without being told mathematically," but she had little to say about how a teacher might help students for whom such discoveries did not come naturally. She did not think that the textbooks she examined were useful in teaching for conceptual knowledge. She criticized textbooks for their focus on getting students to memorize a process rather than understand it. When a text did provide some conceptual explanation, however brief, she generally found the explanation confusing and believed it would not be helpful to students.

Ms. Daniels also had difficulty providing conceptual explanations for the hypothetical problems posed in the baseline interviews or suggesting strategies for teaching the concepts underlying these problems. For example, one problem asked how she might respond to a student who, facing the problem $2.35 \times .7$, says, "I'm confused. This doesn't make any sense. The answers I get are smaller than the numbers I start with. What am I doing wrong?" Her initial responses in all three interviews were statements of the fact that when you multiply by a decimal number less than one, the result is less than what you started with. She was confident in her knowledge of this fact, but uncertain of how to explain why it was true.

Ms. Daniels was then asked to suggest a diagram or story than might help a student understand multiplication by a decimal number less than one. In the first baseline interview, she was unable to construct an appropriate story or diagram. At one point, she struggled to construct a story about money but could not do it.

> Decimals are not used...often. I mean, you use it in money, but this is not a money problem. I mean, how can you say you have \$2.35 and you multiply by \$.70? It doesn't really make much sense.

At another point, she suggested that one might be able to "convert the numbers to fractions and draw figures and shade it,...[but then decided] that would be too complicated for a child to understand."

In the second and third baseline interviews (conducted after Ms. Daniels had completed the mathematics methods course), she was somewhat more successful at providing a story. In both of these interviews, she suggested that a story involving a 70%-off sale of an item costing \$2.35 would be understandable to students and would represent multiplication by a number less than one. She believed that her students would see that "70% is only part of the whole and so it has to be less than the whole, the original price."

But she still had trouble with the diagrams. In the second interview, she began a diagram in which circles represented place values. She recalled the diagram from the mathematics methods course.

> [In] math methods,...we [drew] little circles for ones, tens, and hundredths and showed it that way. But I think that would be a lot more confusing to try to draw a picture than explaining it with a story.

In the third interview, her diagram also was modeled after one presented in the methods course and consisted of a rectangle, subdivided and labeled to indicate the $2.35 \times .7$ as subsections of the length and width. She shaded in the intersections of 2.35 and .7 to indicate the product. She wrote below the diagram the partial products: $0 + 0 + 0$, $0 + 0 + 0$, and $14 + 21 + 35$. Then she decided what place value each product had. After this, she could not complete the explanation. She stated that she thought the diagram should work, but she was not sure of the details, explaining, "I haven't looked at that in a long time," that is, since the methods course.

Ms. Daniels's assertion that the diagrams taught in the methods course would probably be confusing to students seems to be grounded in her own confusion about what they meant and how to use them to represent $2.35 \times .7$ conceptually. The data suggest that this confusion is due to the fact that Ms. Daniels's understanding of the visual representations of decimal multiplication was quite procedural. That is, she had learned how to draw the diagrams for decimal multiplication problems, but she did not understand the conceptual connection between these diagrams and the algorithm.

Ms. Daniels's Views of Her Own Knowledge

Ms. Daniels also thought that she was better prepared to teach for procedural knowledge (arithmetic) than for conceptual knowledge (mathematics). She believed she was "excellent" in arithmetic, and she reported that she had always liked, and achieved well in, computationally oriented courses.

> I consider myself pretty excellent in arithmetic, because I know how to manipulate the numbers and I use the processes a lot. I've had a lot of practice. But math...I don't have the kind of understanding that I would need to have to be excellent in math. Maybe it's because I haven't had a lot of experience in proving theorems and things like that in math.

Summary

In brief, we found that Ms. Daniels was more confident in her arithmetic skills than she was in her conceptual knowledge, that she had difficulty articulating how she would teach for conceptual knowledge, and that she could not complete conceptual explanations for common topics in the elementary and middle school curriculum. As we will see in the next section, this limitation in her knowledge base was apparent in her classroom teaching as well as her responses to hypothetical problems.

MS. DANIELS'S CLASSROOM TEACHING

Teaching for Procedural and Conceptual Knowledge: A Tension Revealed

Observations of Ms. Daniels and interviews about her teaching confirm that she believed in the importance of teaching for both procedural and conceptual knowledge. The value she placed on learning algorithms and procedural skills, and on practicing them until they are "engraved in your brain," is evident in data from all three observation cycles. The majority of the lessons we observed reflected that value. Demonstration of an algorithm and guided and independent practice of that algorithm were central components of the lessons. Ms. Daniels's demonstrations of algorithms included explanations designed to lead to procedural knowledge. For the most part, these explanations were correct and clearly presented.

Also evident in the observation cycle data is Ms. Daniels's concern that students acquire a conceptual knowledge of mathematics, that is, that they learn the conceptual underpinnings of mathematical procedures. She attempted to teach for conceptual knowledge, to some extent, in all three placements. However, the emphasis she placed on conceptual knowledge varied depending on classroom conditions, and she struggled with conceptual explanations in a way she did not with procedural ones. In the next two sections, we reveal a pattern in her teaching for procedural and conceptual knowledge and examine when her attempts to teach for conceptual knowledge were successful.

The Pattern in Ms. Daniels's Teaching for Procedural and Conceptual Knowledge

Although Ms. Daniels believed in the value of teaching for both procedural and conceptual knowledge, several factors appeared to affect her teaching such that she consistently taught for procedural knowledge, but she taught for conceptual knowledge much less frequently. These factors included her knowledge of mathematics and mathematics pedagogy related to the specific topic being addressed, the importance she placed on curriculum coverage, the type of lesson taught, her desire to provide sufficient practice, her perceptions of students' ability levels and interests, and her perceptions of the cooperating teacher's instructional focus.

Limitations in Ms. Daniels's knowledge base led her on some occasions to focus exclusively on procedural knowledge. One strategy that Ms. Daniels used when confronted with limitations in her knowledge base was to provide mnemonics or memory aides to help students remember algorithms rather than conceptual explanations to help them understand. For example, as she explained to the researcher following a lesson on circumference and area of a circle in the fourth placement, she did not know the conceptual explanation for the value of pi or its relevance to the circumference and area of a circle. Lacking that knowledge, she offered mnemonics to help students remember the algorithms. The clues to include pi in the formulas for area and circumference were "...a circle is like a pie...[and pi] is my friend that we always have around when we are working with circles." And, to remember that the radius is used to compute area, "Think about it. Area has an r sound in it. A-r-e-a and radius." Ms. Daniels considered her lack of conceptual knowledge and the associated instructional focus on definitions and algorithms to be problematic. She admitted to the researcher, "I don't just like saying 'Well, this is pi. Remember it,'...[but] where does pi come from? Well, I don't know." Also, regarding the formula for area of a circle, "I just made up my own rule and, this is bad, but I just did not know how else to teach it."

The importance Ms. Daniels placed on covering all the topics in the mathematics curriculum also seemed to work against devoting time and attention to conceptual knowledge. Covering a topic seemed to mean, at a minimum, ensuring that students could correctly perform the relevant mathematical procedures. If Ms. Daniels spent time addressing the conceptual underpinnings of these procedures, she ran the risk of being able to cover fewer topics. Curriculum coverage was particularly salient to Ms. Daniels in her fourth student teaching placement, when she and Mr. Blake (her cooperating teacher) were helping students prepare for the Survey of Basic Skills (SBS), a standardized test that was administered at the end of the school year and focused on procedural skills. On more than one occasion, she expressed a concern that she would not be able to cover the entire course curriculum before the SBS was administered. For example, she said, "I'm concerned that I'm not going to get through the material I was supposed to get through by the time they start testing in a week and a half." The pressure Ms. Daniels felt was particularly apparent during Morning Math, a time set aside by Mr. Blake for reviewing mathematics skills in preparation for the SBS tests. During Morning Math lessons, she seemed determined to cover large amounts of material in a short time.

Another factor that worked against teaching for conceptual knowledge, at least in Morning Math, was that these lessons were defined as reviews. In reviews, Ms. Daniels seemed to assume that the students had already learned the conceptual underpinnings of the mathematical procedures; therefore, all they needed was to be shown or reminded of the procedures and given time to practice them.

Thus a tension emerged in Ms. Daniels's teaching because her own limited conceptual knowledge and her desire to cover the curriculum in the time slots provided and defined by the school interfered with her teaching for conceptual knowledge. This tension was exacerbated by the value she placed on practice. It was not enough simply to introduce (or review) a topic; students must also have sufficient time to practice. As Ms. Daniels explained to students in the Circle (average) group during her fourth placement, following a lesson on circumference and area of a circle,

> Let's try some practice with that, then. We're going to become experts on this stuff yet, so we can make a good grade on that test, on that skills test....You seem to understand it pretty well going over it orally, so you shouldn't have any problems with the worksheet. So: practice makes perfect in mathematics.

Despite Ms. Daniels's tendency to think and act in the classroom in ways that led to procedural teaching, there were some situations in which she clearly attended to the conceptual underpinnings of mathematical procedures in her teaching. For example, she was more inclined to focus on conceptual knowledge in lessons taught to high-ability students and students who expressed an interest in conceptual issues (e.g., through their questions). She believed that higher-ability students, particularly honors students, were more interested than other students in higher-order questions and discovery-oriented activities—instructional strategies typically associated with teaching for conceptual knowledge. She never stated that average or low-ability students should not be taught the conceptual underpinnings of mathematical procedures. However, she clearly indicated, on more than one occasion, the importance of repetition and extensive practice for these students. "Practice is, for the average student, the only way they can learn a lot of times. You know, the more practice they have, the better they're going to be at it." Given the importance she placed on learning procedural skills in preparation for standardized testing, this priority on practice undoubtedly left less time for conceptually oriented lessons.

We also observed a greater emphasis on conceptual knowledge in Ms. Daniels's teaching during her third than her fourth student teaching placement. (The minimal amount of mathematics teaching Ms. Daniels did during her first placement makes comparative statements including that placement inappropriate.) One possible reason for this pattern is the fact that the class we observed in her third placement was an honors class. However, another possibility is that Ms. Daniels's perceptions of differences in the two cooperating teachers' instructional focus and teaching strengths influenced her decisions. That interpretation receives some support from the fact that Ms. Daniels commented about how much Ms. Santo (her third placement cooperating teacher) seemed to know and about her ability to come up with (conceptual) explanations on the spur of the moment. Ms. Daniels looked to Ms. Santo for advice when her own conceptual explanations were unsuccessful, and Ms. Santo's suggestions typically influenced her planning for the following lesson. She did not look to Mr. Blake (fourth placement) for similar input.

To summarize, several personal and contextual factors seemed to exert pressure on Ms. Daniels to bypass teaching for conceptual knowledge. At the same time, in certain situations she did attend to the conceptual underpinnings of mathematical procedures, although never at the expense of thorough treatment of procedural knowledge. Her attempts to teach for conceptual knowledge are examined in more detail next.

Ms. Daniels's Success at Teaching for Conceptual Knowledge

On a number of occasions, Ms. Daniels successfully presented the conceptual underpinnings of mathematical procedures. In each of these cases, her conceptual knowledge and pedagogical content knowledge (Shulman, 1987)[2] about the mathematical topics addressed in the lesson were evident

[2]In theory, content knowledge and pedagogical content knowledge are distinct knowledge domains. However, in practice, the boundaries between these domains are fuzzy (Marks, 1990). In part because of these fuzzy boundaries and in part because of limitations in our data, when analyzing participants' classroom teaching, we often found that we could not clearly determine whether their decisions and actions were influenced by their content knowledge, pedagogical content knowledge, or a combination of the two. When the distinctions seemed unclear (typically in situations where a participant's teaching was problematic in some way), we identified the problematic aspects of the participant's teaching and suggested possible limitations in content knowledge or pedagogical content knowledge that might account for the problems.

in her teaching and in the interviews before and after instruction. With respect to pedagogical content knowledge, the interviews revealed that Ms. Daniels considered concrete and visual representations to be important for conceptual learning. Further, she typically accompanied her explanations with representations or practical applications. The success of her attempts to teach for conceptual knowledge seemed to be directly related to the power of those representations (e.g., their ability to make the content comprehensible, their appropriateness to the abilities and interests of learners; Shulman, 1987). A good example is her introduction of volume to the Rectangle (above average) group during her fourth placement. Ms. Daniels began the introduction by comparing volume to surface area, explaining that surface area is "the distance around the outside of a three-dimensional figure." Volume is "the space inside of...a box, a rectangular prism." She then showed the pupils an empty cardboard box, which they identified as a rectangular prism. She explained, "[It's a] rectangular prism. And it just so happens that this rectangular prism is filled with cubes or cubic units." Ms. Daniels held up a small wooden cube and said,

> So, we can call it a cubic unit. Now, what I would like for you to do, I need a volunteer. OK, [Janice], I want you to somehow count how many cubic units cover the volume or the inside of this box. Do that now. Somehow figure it out. If you have to, dump them out and count them.

Ms. Daniels left the pupils to solve the problem on their own while she worked with the Circle group. She returned periodically during the class session to check on their progress and to offer suggestions for how to approach the problem. For example, when several students were counting the cubes and getting different numbers, she suggested, "Why don't you give ten to each person and see how many tens you've got." When she returned for the final time, the pupils had agreed that the correct solution was 90 cubic units. She asked, "Okay, now do you think there is an easier way to do this?" and told the students, "See if you can figure out what the pattern is." She then led a discussion in which the pupils shared their solutions and developed the formula for volume of a rectangular prism. As they explained, they first computed the number of cubes in one layer and then figured out the number of layers. From there, they were able to calculate the number of cubes that the box would hold. Ms. Daniels noted, "That's exactly right. There is a formula for volume. You take the length times the width times the depth or the height is what they call it. L times W times H." She concluded the lesson by writing the formula on the board.

This lesson provides an example of Ms. Daniels's use of powerful representations to emphasize the conceptual underpinnings of mathematical algorithms. However, although this was one of her strongest attempts to teach for conceptual knowledge that we observed, it is not as strong as it might have been. In concluding the lesson, Ms. Daniels jumped rather quickly from the concrete representation to the formula. She did not make an explicit connection between the solution to the problem and the formula. Nor did she explain that the problem is an example of a representation or model that leads to the derivation of the formula, *rather than just verifying* the solution obtained with the formula.[3]

On other occasions, Ms. Daniels attempted to provide conceptual explanations, but the explanations only served to confuse her students. Difficulties in explaining several concepts in the unit on number theory (e.g., prime and composite numbers, counting numbers) seemed to be related to her own lack of clarity regarding those concepts or how to teach them. Her explanation for determining the number of primes that must be checked to see whether 113 is prime or composite is illustrative. The lesson segment reproduced below followed a class discussion and agreement that the numbers 2 through 10 are not factors of 113.

> We know that 11 doesn't go into 113 and the next number that 11 goes into is 121. So we can just stop right there, right? Because 113 is the closest you're going to come to 121. 12 times 12 is 144, so we're still going to be going past the 113, right? See, 10 times 10 is 100. 11 times 11 is 121. And 113 falls right in between there. So when we get to 11, we know that 11 times 11 is greater than that so that is where we stop testing our divisibility.

Ms. Daniels's attempts to teach for conceptual knowledge also were less successful when she was unable to come up with powerful applications or representations. One example was her explanation of prime factors, which used the representation of a factor tree with branches and leaves:

> And the way we determine these prime factors, we kind of sift them out. We use what we call a factor tree. OK? We say, well, two factors of 12 are 2 and 6. And 2 is already prime so you can't branch out any further. But you know that 6 has some prime factors in it or some other factors. So the numbers that are still hanging on the ends of these branches of your tree, these are just like little leaves here, are going to be your prime factors.

The procedure of creating branches and leaves did not develop conceptual knowledge. In fact, the factor tree representation functioned as a visualized procedure, rather like a mnemonic for determining the prime factorization of a number. In that respect, it was similar to the mnemonics Ms. Daniels used to help students remember the formulas for circumference and area of a circle.

Ms. Daniels usually realized when students were confused by her attempts to provide conceptual explanations. On several occasions in her third placement, she sought the advice of Ms. Santo and, on the basis of that advice, attempted another explanation on the following day. In some instances, the second explanation was of much higher quality. One example is Ms. Daniels's attempt to explain why zero is not a counting number. Her first explanation was in response to a student's question:

> *Ms. Daniels:* When you learned how to count, what did you start with?
>
> *Brad:* One.
>
> *Ms. Daniels*: One. One is the first counting number because it's countable. Zero. You can't count zero. A countable number is what you can count by seeing it: 1, 2, 3, 4, 5....You have to have something to be able to count....If you had nothing sitting on a table and they said, 'count how many oranges are on the table,' you can't count anything because there's nothing there.

Ms. Daniels sought Ms. Santo's advice after the lesson, and she was given the suggestion to use an example of a counting situation familiar to the students such as counting the pupils in the class. Her explanation on the following day consisted of a clear, concise, contextual example:

You can't count zero. Like if I wanted to count the people in the room. If I was starting with Jackson here, I wouldn't say 0, 1, 2, 3. Right? Would that give me the right answer?

(It should be noted in this context—and we return to this issue later in the paper—that although Ms. Santo provided some concrete examples that Ms. Daniels used to construct better conceptual explanations, Ms. Santo's own approach to teaching for conceptual knowledge was generally abstract.)

Understanding the Tension: A Final Example

Our final example is taken from a lesson on division of fractions that occurred during Morning Math in Ms. Daniels's fourth student teaching placement. Ms. Daniels demonstrated and provided a procedural explanation for the division of fractions algorithm. Then, in response to a student's request for a conceptual explanation for the algorithm, she attempted to provide a concrete example and accompanying diagram. The example was, What portion of a wall would you be able to paint if 1/4 of the wall were painted and you had enough paint to cover 1/2 of the remaining area? Ms. Daniels realized partway through solving the problem that it was an example of multiplication of fractions rather than division. She abandoned her attempt to provide a concrete example and, for the remainder of the lesson, focused on computational procedures for division of fractions and related topics. She did not return to the explanation of division of fractions in a subsequent lesson. In discussing the lesson with the researcher later that day, Ms. Daniels stated that "the explanation...wasn't very good. But I think by the end of the time, that they had picked up on it." Her major concern was that "I just spent too much time on it. I mean, as a result, I had to cut short my other lessons...." She also reported that her cooperating teacher's feedback was that the lesson was too long; he told her she should have cut it off after 20 minutes. (See Borko et al., 1992—where this lesson is featured—for a more detailed description.)

[3]Another notable characteristic of this episode is the students' participation as mathematical problem solvers. Such active involvement certainly is in keeping with the curricular goals and instructional strategies related to teaching for conceptual knowledge that are recommended by the National Council of Teachers of Mathematics (NCTM). Unfortunately, there were too few examples of this type of student participation in our observations of Ms. Daniels's teaching to enable us to draw conclusions about its role in her attempts to teach for conceptual knowledge.

A comparison of this episode with characteristics of Ms. Daniels's successful attempts to teach for conceptual knowledge provides a summary of factors that contributed to the tension between procedural and conceptual knowledge that characterized her teaching. Turning first to the question of when Ms. Daniels attempted to teach for conceptual knowledge, only one of the factors generally associated with teaching for conceptual knowledge was present—Ms. Daniels's perception of students' interests. Looking at the other factors, the lesson was a review lesson; it took place during Morning Math when Ms. Daniels felt compelled to keep lessons short and to cover specific topics in the curriculum; she seemed to correctly perceive Mr. Blake's intention that Morning Math focus on procedural skills; and her relevant knowledge base was limited. Apparently, Ms. Daniels's perception of student interest was sufficient reason for her to attempt the conceptual explanation, but the counterbalancing factors overwhelmed the attempt. Her ideas about how the time should best be spent and what her cooperating teacher would be most concerned about, coupled with her limited conceptual knowledge, seemed to lead her to abandon the attempt to teach for conceptual knowledge.

It is important to note that Ms. Daniels did not attempt to improve on her explanation in a subsequent lesson. This reaction contrasts with her responses on several occasions in the unit on number theory (Placement 3), when she experienced difficulty with conceptual explanations. In that context, when most other factors supported a decision to teach for conceptual knowledge and when Ms. Daniels felt that she could turn to her cooperating teacher for assistance, she consistently attempted to improve on initially shaky conceptual explanations.

Thus we find an unresolved tension in Ms. Daniels's teaching between her desire to teach for both conceptual and procedural knowledge, on the one hand, and pressures exerted on or by her, many of which supported procedural teaching primarily, on the other hand. This tension, which led Ms. Daniels to teach more consistently for procedural knowledge, was more situational than it was personal. On a personal level, she did not always have the content or pedagogical knowledge she needed to teach for conceptual knowledge, despite her desire to do so. However, situational factors seemed to determine when she had or created the opportunity to learn that knowledge. Pressures to prepare students for tests, to cover designated topics in the curriculum, and to use school time for review and practice of procedural skills were obstacles around which Ms. Daniels (often inadvertently) found little room to implement her stated intention to teach for conceptual knowledge. In contrast, when situational supports for teaching for conceptual knowledge were in place—for example, a cooperating teacher who was a resource for such teaching, or students who asked conceptual questions, or (hypothetically) time in the school schedule to take a break from the pressures of preparing students for tests—Ms. Daniels took advantage of opportunities to learn more about teaching for conceptual knowledge. Unfortunately, as the next sections illustrate, her opportunities in both the teacher education program and the placement schools were limited by similar unresolved tensions.

TENSIONS OVER CONCEPTUAL AND PROCEDURAL KNOWLEDGE IN THE UNIVERSITY TEACHER EDUCATION PROGRAM

Tensions in the Mathematics Methods Course

The instructor's commitments. As the semester began, we interviewed the mathematics methods course instructor to determine his goals for the course. He had given careful consideration to what he would cover during the course and how he would do it. He had two primary goals. First, he intended to emphasize the conceptual aspects of teaching and learning mathematics, specifically the relationships between procedures and concepts and among concepts. About this, the instructor said,

> I try to get them to understand the relationship between the various parts of the curriculum, so [for example] we will emphasize relationships between whole numbers and rational numbers...from a pedagogical perspective particularly....In addition, we will also look at the operations and try to show the relationships between addition and subtraction and multiplication and division,...[and] try to understand how we can use those relationships in helping children understand mathematical concepts.

The instructor's second goal was to provide the student teachers with classroom "survival strategies," that is, strategies they could use to teach conceptual knowledge, while also satisfying the schools' demands that procedures be mastered. For example, the instructor wanted the student teachers, and the students they would teach, to think about algorithms in terms of underlying concepts, not just in terms of computations.

> In the area of computation work, I would like to ignore a lot of the computation work because in my own value scheme it is not very important any more. [But] I have to be a realist, and I know that every textbook out there has addition and subtraction and multiplication and division of whole numbers and rational numbers—the algorithms. So, I want [the student teachers and their students] to understand those algorithms. [And] I want them to know how to use manipulatives to teach those algorithms. My objective then will be that they have those understandings.

The instructor knew that there were entrenched tensions between his goals and those of the schools (see also Underhill, 1991, for more extensive coverage of the various tensions or dilemmas faced by the mathematics methods instructor in the study). And he knew the tensions would make the course difficult to teach. He realized that what he wanted to stress—the conceptual aspects of elementary and middle school mathematics using a Piagetian developmental focus and stressing concrete operational thinking—would not be the emphasis the student teachers were likely to encounter in their placement schools. He also knew that it would not be the emphasis that the student teachers were already familiar with, and probably expected, from their own experiences learning mathematics. Despite the tensions, the instructor believed that he could not, in good conscience, omit either focus from the course he was about to teach. (We expect that many teacher educators today find themselves facing similar tensions as they contemplate the design of their methods courses.) We found, however, as the course unfolded, that the instructor's efforts to teach for conceptual knowledge (with a concrete operational focus) tended to be undercut. This outcome was due in part to the instructor's decisions and actions during the methods course and in part to the student teachers' collective response to the course and to the teacher education program as a whole.

The instructor's actions. Beginning with the first class, the instructor divided the class meetings into two parts: (a) whole class minilectures and discussions, and (b) laboratory experiences in which the students worked in small groups on problems related to some of the topics of the whole class session. During the whole class session, the instructor routinely introduced a topic, for example, two-digit addition, and then gave a demonstration of how he would teach it to "help learners understand math." The student teachers then worked on the same topic in their small groups, using the story formats, manipulatives, or diagrams modeled by the instructor. In other words, in each class session the instructor offered demonstrations of what to do in the classroom—demonstrations that he believed would develop both the children's computational skills and their conceptual knowledge; then he asked the student teachers to practice them.

All the instructor's demonstrations were designed to match a sequence of developmental levels for teaching a mathematical idea. In the initial interview, the instructor described the levels:

> When I teach mathematics, I...talk...in terms of four developmental levels of teaching a mathematical idea. The first one is the "readiness level," in which I focus on using manipulatives in problems, but not writing anything down. Then at the "concrete level," you do the same thing with the manipulatives [and write the problem down]. Then with the "semi-concrete [level]," you give up the manipulatives and go to something that is a visual—pictures and diagrams to represent the solutions to problems. And then finally, you have the abstract level where you are just dealing with formal mathematical symbolism. But the way I try to get them to see it, is that we don't want to go to that next step until we know that the child has iconic imagery that he or she can use to support [the symbols] he or she is writing down on the paper.

In the fourth observation, the instructor discussed the topic of two-digit addition by demonstrating how it could be presented as a readiness experience (tell a story using the numbers to be added and demonstrate with Cuisenaire rods or place value blocks), and then as a concrete experience (use the rods or blocks together with the story and the written, symbolic form of the problem). Afterwards, he sent the student teachers to small groups with the following advice about developing their own concrete experiences for two-digit addition problems:

> The thing that we have to work on the most...is that there's something happening with these rods [and there's] also something happening in the problem. The purpose of going through this is to make connections between what's happening here [with the rods] and what's happening here [with the symbols]. [The instructor reminds the student teachers how he worked each step with the rods and then recorded it with the symbols.] So what we're trying to do at each step...is to say that everything we do here [with the rods] is reflected with something that happens here [with the symbols].

On numerous occasions, the instructor told the student teachers that using readiness, concrete, and semiconcrete experiences would help children develop a conceptual understanding of mathematics. By joining the three levels with Bruner's (1963) enactive and iconic ways of knowing, the instructor explained that the three levels help children know mathematics kinesthetically (enactive) and through mental imagery (iconic). Later, when learners have abstract or symbolic experiences (when there are no manipulatives or diagrams available), they can call on their enactive and iconic knowledge to make sense of the symbols.

Despite the instructor's intentions, analysis of the 16 observations of the mathematics methods class sessions suggests that many of the student teachers perceived the demonstrations and practice sessions as routines to memorize, rather than explanations to understand. To some extent, this tendency was encouraged by the instructor's step-by-step way of demonstrating mathematical activities, as well as his commitment to a particular organization and sequence of mathematical experiences, and his requirement that the student teachers practice and be tested on what he modeled. For example, the instructor demonstrated the four developmental levels for each mathematical operation covered in the course (addition, subtraction, multiplication, and division), and in doing so, he always included (explicitly) linguistic scripts or stories (for readiness experiences), use of Cuisenaire rods or paper folding/shading (for concrete experiences), and diagrams (for semiconcrete experiences). The student teachers, in turn, were asked to practice the same steps and strategies and to use them in answering questions on quizzes and the midterm. As a result, the demonstrations and practices could appear to focus on step-by-step procedures for teaching.

In addition, only occasionally did the instructor include explicit conceptual explanations for the link between, for example, a paper-tearing activity and a division algorithm, and when he did so, the student teachers tried to memorize the explanation or said they did not understand it (see Borko et al., 1992, for a detailed examination of the instructor's explanation of the division of fractions algorithm and the student teachers' reactions to it).

The student teachers' questions and comments about the demonstrations almost always focused on the details of specific procedures. In their small groups and on quizzes and the midterm, they treated the four levels, with the components listed above, as formulas that they were supposed to identify and apply. In the fourth and fifth observations, for example, with the midterm looming ahead, the student teachers asked a series of questions about how they should set up learning experiences for three-digit multiplication for each of the four levels. After some attempt to repeat the process he had previously demonstrated, the instructor expressed dismay that the student teachers seemed to be trying simply to memorize activities to be used at each of the four levels.

Thus, as the student teachers' questions reveal, the focus on procedures was not only a function of what the instructor did or did not do. Some of the tendency to focus on procedural details came from the student teachers. They had their own tensions to resolve as they proceeded through the mathematics methods course and the teacher education program, and the way they tried to resolve the tensions led them to focus on procedural knowledge.

Tensions in the Student Teachers' Experience

Tensions felt by the student teachers derived from competing pressures to balance the demands of their university course work with those of their placement schools. On the one hand, they had to meet their university professors' expectations for course work; on the other, they had to meet daily responsibilities at their placement schools. (These competing expectations and the tensions they created for the student teachers are described in more detail in Eisenhart, Behm, & Romagnano, 1991.) Given the two sets of demands and the fact that the student teachers had teaching responsibilities every day, they did not have time to develop their own set of classroom activities. Instead, they tried to adopt the activities they learned about in their course work. In their words, they needed "ideas that will work"—activities that could be transported, with little modification, into their own classrooms. Thus, from course work, such as the mathematics methods course, they found it most useful to know exactly how to do an activity and for whom it would be appropriate. Under time pressure to learn the activity and given limited time in each methods class, the student teachers tended to focus on procedural details of the activities they were presented (i.e., how to implement activities in the classroom) and either assumed or ignored the conceptual aspects (i.e., the meaning behind, or the mathematical reasoning for, doing the activities).

By the 10th observation, the tension became increasingly evident in the mathematics methods course as the demands on the student teachers from their placement schools escalated. The student teachers worried that they spent too much time in the course on things that would not help them teach. For example, after the instructor suggested that part of the value of having the student teachers practice in small groups was to facilitate individuals' own conceptual knowledge, one student explained that facilitation was fine, but "we were never told for sure what was right or what was wrong," and "we were not presented with enough good concrete examples"; the student teacher did not think the answers or examples provided in groups were good, because they came from other novices. The instructor called for patience, saying "people are growing in these concepts very slowly," and he applauded their progress, but the student teachers were not satisfied. One responded, " But I need to know it now...exactly." In the same discussion, another student teacher asked the instructor to provide more "lists" and more "concrete things to refer...to." This student and several others were especially upset because their class notes were not developing into a resource they thought they could use to teach their own students later. The instructor pointed out that the textbook assigned for the course was such a resource. Some of the students responded that they did not have time to read the textbook and indicated that since they must attend class, the resources should be provided there.

The students also resisted some of the instructor's attempts to get them to think conceptually. For example, on several occasions (including the midterm), the instructor explicitly told the students not to use formulas, for example, in finding the area of a triangle. Several students used the formulas anyway and accepted the negative consequences.

Summary

In summary, the tensions felt by the student teachers seemed to blind them to the conceptual knowledge, and ways of teaching for conceptual knowledge, that the instructor was trying to help them learn. At the same time, tensions felt by the instructor obscured some of the ways in which his teaching emphasized procedural routines, rather than conceptual knowledge. In the instructor's case, his response to the tension of teaching for conceptual knowledge and teaching strategies that matched the school's goals was to demonstrate and sequence teaching activities in ways that he believed would lead students to understand the conceptual underpinnings of mathematical procedures. The collective response of the student teachers to the tension created by competing demands from the university and placement schools was to focus on their university courses as a source of activities that could be transported directly to their teaching classrooms and to put pressure on instructors to modify their courses to achieve that end. The product of the confrontation between the two resolutions was (among other things) an emphasis on procedural, rather than conceptual, knowledge.

PROCEDURAL AND CONCEPTUAL KNOWLEDGE IN THE SCHOOLS

In this section of the article, we examine the messages about teaching for procedural and conceptual knowledge that originated at the central district level and then were interpreted in the schools and classrooms we studied. Our analysis yielded a picture of competing messages that were interpreted somewhat differently at each level.

The District Level

Perhaps unwittingly, the central administrators in charge of the mathematics program pushed for pedagogical outcomes that were in conflict. On the one hand, they mandated a curriculum and testing program that emphasized procedural knowledge as reflected in the Standards of Learning (SOLs) mandated by the state. Examples of objectives in the SOLs include the following: (Grade 5) The student will find the product of two numbers expressed as decimals such that the product contains no more than three decimal places; (Grade 6) the student will multiply with simple fractions having denominators of 10 or less; (Grade 8) the student will add, subtract, multiply, and divide with fractions and mixed numerals. Procedural knowledge was also emphasized in basal series for mathematics and the computerized evaluation system that accompanied it. For example, in the first lesson on multiplication of fractions in the fifth grade text, the students were shown how to multiply numerators and denominators, and in the lesson on multiplication of decimal numbers, they were shown how to count decimal places. Similarly, in the sixth grade text students were shown how to invert and multiply when dividing fractions after a few sentences about inverse operations and reciprocals.

On the other hand, these administrators expressed a verbal commitment to teaching for conceptual knowledge. For example, when the mathematics supervisor was asked if the school district had policies or guidelines about how mathematics should be taught, she responded,

> Not formally, I suppose....[But there's] a point of view that it will be highly manipulative and open-ended..., encouraging creativity and problem solving....Don't insist that rote procedures, for example, have to be followed just the way that you would do the problem if you were doing it....When I talk with teachers...I think what they hear is that math lessons should be fun...interesting...a lot of interaction, student to student, and student to teacher....[The students] need to verbalize,...they need models in front of them.

The district also offered an in-service program that was intended to help teachers develop strategies for teaching for conceptual knowledge.

The two messages were repeatedly illustrated in the words of the administrators. For example, at one point the associate superintendent stated,

> Both the standardized testing and our local management system help point out weak spots [relative to the SOL objectives], individually and by class, so that we [can] target instruction and testing, so that we'll....have a net improvement in our student performance.

Later in the same interview, the associate superintendent talked about how important it was to teach for conceptual knowledge through applications, manipulatives, and modeling. He stressed: "I would like to see more modeling than we typically have....I would like for us to have more modeling and be closer to the concrete kind of an orientation [as] opposed to just a chalk and chalkboard." In accord with this goal, he pointed out that the fifth and sixth grade in-service program stressed: "things like the use of geometric models, area, volume, perimeter, and Cuisenaire rods as a flexible approach to a number of things, including fractions and decimals."

In brief, we found district emphasis on procedural knowledge reinforced through a formal evaluation system that encompassed the state SOLs, a standardized Survey of Basic Skills (SBS), literacy tests, and basal chapter tests. In other words, there was institutional accountability for procedural learning. In contrast, messages about teaching for conceptual knowledge were communicated informally, for the most part, and through in-service activities focusing on the use of manipulatives and problem solving. The district administrators believed in conceptual learning but offered only encouragement for developing it.

The Placement Schools

The two messages communicated by the district administrators were also heard at the schools where Ms. Daniels was placed for student teaching. In some cases, school-level personnel expressed strong commitments to teaching for conceptual knowledge, as well as for the procedural knowledge needed for tests. However, there were few clear models of how, in practice, to teach for conceptual knowledge or how to balance the demands to prepare students for tests and to develop their conceptual knowledge.

Ms. Daniels's first placement. The principal at Ms. Daniels's first school was a strong instructional leader with commitments to learner involvement and innovative instructional strategies. Her views about mathematics teaching were similar to the central administrators'. On the one hand, she stressed the importance of satisfactory performance on the basal and other tests. On the other hand, she was concerned about developing conceptual knowledge.

> We use test results. We use teacher judgement, [and] the informal test materials, and, then, of course, we use our basals. But I discourage just using the basals. I love using everything else, you know, to involve students. I'm real up on innovative techniques and hands-on activities, things to help our students. We encourage groupings within groupings and hands-on approaches and use of manipulatives.

She had purchased quite a few manipulatives the previous year, and she made repeated references to the importance of problem-solving activities, using manipulatives, learner involvement, and the development of conceptual knowledge. Although she said that some teachers in her school did not like to use groupings and manipulatives, she said that she insisted they be used.

The principal identified Ms. Bender, Ms. Daniels's cooperating teacher, as one teacher who agreed with her position. About Ms. Bender, the principal said,

> She's very knowledgeable of hands-on activities. Well, she sets the tone by stating objectives and making kids aware of the objectives. But she uses a lot of manipulatives and small group or peer group kinds of activities. And she does the lecture method, modeling and working with the students, giving individual assistance as needed. But she does, also, do a lot of hands-on things.

Ms. Bender, herself, was keenly aware of the prominence of the testing program. She said, "It seems like we're testing all the time!" And she worked hard to assure that her lessons would prepare students for the tests: "We have a lot of practice, especially with multiplication tables." She used games and worksheets involving mathematical skills to supplement the basal text.

Ms. Bender also wanted her students to develop conceptual knowledge of mathematics, but her conception of conceptual knowledge was different from her administrators', especially related to topics new at grade 6. When questioned about the important aspects of teaching division of fractions, Ms. Bender said,

> I think the basic concept—that I expect them to know if they're going to be doing division of fractions—is the concept of reciprocals and the inverse operations of multiplication and division. If they don't understand that, then there's no use of me...teaching it the same way my teacher taught it to me: "This is the way it's done. Do it."

In this statement, Ms. Bender expresses a view of mathematics pedagogy in which conceptual knowledge is a goal, but the means to achieving this goal are abstract rather than concrete. Her classroom teaching reflected this emphasis. When the interviewer reflected back on his interview with Ms. Bender, he pointed out,

> She doesn't talk about using any pictures or manipulatives in teaching. Her...emphasis appears to be on reciprocals and getting the right algorithm down, and there seems to be very little that one would call development of understanding from a concrete or semi-concrete perspective.

And in describing Ms. Bender, Ms. Daniels reported that "[Ms. Bender] had tons of books...that she gets activities and ideas from...[but]I've never seen anything like Cuisenaire rods used or anything...."

In brief, both the principal and Ms. Bender acknowledged the importance of the testing program and students' performance on tests. However, they had disparate views on how to develop conceptual knowledge among learners of this age. Whereas the school principal shared with the district administrators and the university mathematics methods course instructor a commitment to developing conceptual knowledge through concrete and semiconcrete experiences, there was no evidence that either Ms. Bender or her students used manipulatives during mathematics lessons. The evidence we collected indicates that she taught more advanced topics either for procedural knowledge or for formal, abstract understanding.

Ms. Daniels's third placement. The principal at Ms. Daniels's third placement, a junior high school, expressed views similar to the first principal. However, Ms. Santo, who was Ms. Daniels's cooperating teacher during the third placement and also the mathematics department head, believed that only traditional, that is, either procedural or conceptual-knowledge-as-abstraction, teaching was occurring at the school. Ms. Santo was aware that some administrators encouraged the mathematics teachers to teach for a different type of conceptual knowledge. For example, she noted that,

> ...our principal challenged us at the beginning of the year, for all of the mathematics classes, to have one hands-on activity, you know, per week with their math classes. And I know that as a department we...really need to work on that....[But] we don't do that as much as we should.

As was found in Ms. Bender's case, the researcher who observed Ms. Santo's class reported that she taught for conceptual knowledge from an abstract, rather than a concrete, orientation:

> She is quite competent in math and has a good understanding of the subject matter. She doesn't go in for manipulatives much. She thinks of herself as a good traditional teacher. This is a pretty good description. She pushes students for understanding but nearly always from an abstract, symbolic perspective rather than through the use of manipulatives or pictures and diagrams.

And Ms. Daniels said:

> Other than a game and Möbius strips once each, she doesn't do that many activities. She just does homework and word problems. She has puzzles for that class a lot, brain-teaser type things....They just go by the book pretty much.

In brief, Ms. Santo was a traditional mathematics teacher with a strong mathematics background. She taught for conceptual knowledge through explanations and problem solving aimed at abstract or symbolic ways of knowing. She was aware of messages from the central administration and the principal to use manipulatives, but she did not change her teaching practice to address those messages.

Ms. Daniels's fourth placement. The principal in Ms. Daniels's fourth placement school had a distinctive interpretation of central administration's views. He said,

> The school-wide priorities [for mathematics] are determined in two ways. One, the district has a priority on hands-on manipulative mathematics in the primary grades....After second grade, the [priority is] the results of pre- and post-testing.

Mr. Blake, Ms. Daniels' cooperating teacher, also focused on the role of testing in mathematics instruction. For example, he told us,

> The expectations for teachers are, you know, very high....teachers [have become very] conscious of test scores...what they [the teachers] will teach during the course of the year, the areas of math, would probably come from the SOLs and the areas of...the pre-tests and post-tests.

Mr. Blake also talked about the value of manipulatives in teaching. "Whenever possible I like for them [the students] to manipulate, to use things, to see how they work." However, his responses to hypothetical classroom scenarios painted a different picture. For example, when asked to respond to a student's difficulties with division of decimals, he stated,

> My philosophy of teaching math is, "Ok, so you can work a computation problem? Working a computation problem is easy, but yet, what are your steps that you go through in the process of getting or arriving at your answer?"

Also he told us, "I would grade [a quiz] for accuracy. Just checking the answers, not the process." When asked what he thought was the basic concept in a decimal division scenario, he replied "Moving the decimals."

Regarding Mr. Blake's use of manipulatives, the researcher wrote in his journal:

> [Mr. Blake] talked about using manipulatives a lot, but as far as I could tell, he didn't use them for rational numbers. And when I tried to get him to talk about using them with rational numbers, he made up an example that he couldn't seem to solve...I got the definite impression that even though he talks about kids' understanding, that what he really means is kids remembering procedures.

Ms. Daniels's comments about Mr. Blake's classroom focused on the testing pressure, too. For example, when asked about her reactions to that classroom, she said, "I am running out of time, I feel like, because we have so much to get in before these tests start, the SBS test in May." Also, when asked how decisions are made about mathematics content, she replied,

> You know, SOLs and the math book kind of tells you when to test them on it. There are tests at certain times of the year that, you know—sometimes you have to teach for the test, which is kind of bad, but you want your students to do as well as they possibly can.

Mr. Blake's feedback to Ms. Daniels about her teaching provides additional evidence regarding priorities in her classroom. For example, regarding the division of fractions review episode, Mr. Blake expressed concern about Ms. Daniels's approach and her allocation of time. According to Mr. Blake, Ms. Daniels saw the goal of remediation as "They have to understand this before I quit." But he said, "No. It's remediation. Remediation today, tomorrow, the next day, and the next day, and the fifth day they'll get it." Also, he felt that she should have stopped the lesson much sooner. He mentioned nothing about her inaccurate representation.

In summary, both the principal and Mr. Blake seemed to organize their expectations around the testing program which emphasized procedural knowledge. Although Mr. Blake talked about manipulatives, we have no evidence from Ms. Daniels or from our observations that he used them in his mathematics instruction. We have quite a lot of evidence that he focused on procedural knowledge, and we know that on at least one occasion he disagreed with Ms. Daniels's effort to teach for understanding.

Summary

As the preceding analysis indicates, the district's central administrators and principals pushed for two pedagogical outcomes that were in conflict, and they held the teachers accountable for one but not the other. Teachers were accountable for their students' performance on standardized tests and for a management system that emphasized procedural knowledge. At the same time, the administrators encouraged teachers to use—but did not hold them accountable for—a form of teaching for conceptual knowledge (with a focus on concrete and semiconcrete representations) that also was encouraged by the university mathematics methods course instructor.

The cooperating teachers at Ms. Daniels's placement schools were aware of the two pedagogical goals and tended to resolve the tension between them by stressing procedural knowledge and ignoring the call to teach for conceptual knowledge using concrete and semiconcrete representations. Ms. Bender, Ms. Daniels's first cooperating teacher, rarely used manipulatives, pictures, or diagrams for simple mathematics and emphasized procedural knowledge and abstract conceptual explanations for advanced topics. Ms. Daniels's second cooperating teacher in the study, Ms. Santo, also taught for conceptual knowledge but at the abstract rather than the concrete or semiconcrete levels. Her third cooperating teacher, Mr. Blake, approached virtually all topics in the mathematics curriculum as procedures to remember and perform efficiently; he seemed to have operationalized almost no commitment to the development of conceptual knowledge.

Thus, in the three classrooms, Ms. Daniels had very few opportunities to observe teaching for conceptual knowledge as characterized by the school administrators and her mathematics methods course instructor, nor did she receive encouragement for trying it. On the other hand, she had numerous opportunities to observe teaching for procedural knowledge, and she received encouragement for doing it. The teaching opportunities available to Ms. Daniels in the placement schools favored teaching for procedural rather than conceptual knowledge, at least as teaching for conceptual knowledge was envisioned by the school district administrators and the university teacher education program. As we also found in Ms. Daniels's teaching and the teacher education program, teaching for conceptual knowledge was falling through the cracks at the placement schools.

CONCLUSION

Like most members of the mathematics education community, Ms. Daniels believed in the importance of teaching mathematics for understanding and in the need to teach for both procedural and conceptual knowledge in order to achieve understanding. However, her knowledge of both content and pedagogy limited her ability to articulate how she would teach for conceptual knowledge, and she actually taught for conceptual knowledge only rarely. When we examined her classroom teaching, we found, on a personal level, that she was more successful with, and more confident of, her own procedural knowledge base. Situationally, she faced and agreed with a need to prepare students for skills-oriented tests, cover the designated skills-oriented curriculum, and provide time for students to practice procedural skills—all commitments of time and energy that came first, both in her own mind and in the minds and practices of many around her. She made some attempts to teach for conceptual knowledge and was sometimes even quite persistent in her search for a conceptually based explanation. However, most often the time commitments to procedural knowledge, coupled with personal factors, interfered with both her opportunities and her attempts to teach for conceptual knowledge.

Further, Ms. Daniels and the other student teachers developed a stronger commitment to teaching for procedural knowledge than conceptual knowledge. Despite the methods course instructor's efforts to demonstrate teaching for conceptual knowledge, the student teachers tended to interpret the information he provided in procedural terms, as routines for, or lists of, pedagogical strategies. In Ms. Daniels's case, the instructor's efforts did meet with some limited success. She tried to incorporate into her teaching some of the stories, concrete and semiconcrete representations, and other activities she had learned in the methods course. Unfortunately, because she seemed to focus on remembering the procedures for these activities but not their conceptual underpinnings, she often remembered incorrectly or only partially, and she typically did not draw connections between representations and algorithms in her teaching. Thus, although she did transport some of the ideas from the methods class into her teaching, these ideas were not used optimally to teach for conceptual knowledge.

The public schools in which Ms. Daniels was placed for student teaching did almost nothing to reinforce the ideas she learned in her methods course. She had few opportunities to observe teaching for conceptual knowledge, and the opportunities she did have focused on abstract rather than concrete or semiconcrete representations. Further, she had little encouragement to develop the kind of teaching for conceptual knowledge that was stressed by the methods course instructor and the district-level administrators. Instead, she found support and some pressure to teach procedures and to drill her students on the procedural aspects of mathematics.

The outcomes we found did not develop without some kind of struggle. Most of the key actors—the student teachers, the methods course instructor, the district-level personnel, and many of the cooperating teachers—grappled with how to implement their commitment to teaching for conceptual knowledge and how to counteract the schools' over-emphasis on rote acquisition of procedural knowledge. In Ms. Daniels's case, she seemed to have been affected by an unacknowledged conflict between her own stated desire to teach for both procedural and conceptual knowledge and pressures to teach procedural skills that she found compelling. The mathematics methods course instructor clearly recognized a tension between his conceptually oriented goals for the student teachers in his class and the procedurally oriented goals that the student teachers would encounter in their placements.

As a group the student teachers in Ms. Daniels's cohort felt the tension identified by the methods course instructor: They were torn between pressures to meet the sometimes competing but always demanding expectations of the university, on the one hand, and their placement schools, on the other. Finally, within the school district, administrators, principals, and cooperating teachers pushed for procedural and conceptual pedagogical outcomes that were in conflict. These competing tensions created a context in which teaching for conceptual knowledge remained an unrealized goal for teachers, teacher educators, and student teachers alike.

As we have hinted throughout this paper, the tensions were not created by the individuals alone. They were (and continue to be) created by a system of mathematics education with various requirements, traditions, and constituencies that exert competing pressures on any mathematics teacher educator or student teacher. Individuals in teacher education programs must respond to these competing-pressures-creating-tensions in some way. In our study, the resolutions led to an outcome in which teaching for conceptual knowledge tended to fall through the cracks.

This outcome is not one that would be favored by the mathematics education community today. The community's position is that whenever possible, "...mathematical reasoning, problem solving, communication, and connections must be central....Computational algorithms, the manipulation of expressions, and paper-and-pencil drill must no longer dominate school mathematics" (NCTM, 1991, p. 19).

Given the importance of the routines and patterns established during a teacher's beginning years (Feiman-Nemser, 1983), it seems important to ask what must happen to increase the likelihood that student teachers will have the opportunity to teach and learn to teach for conceptual knowledge in accord with the mathematics education reform agenda. That is the question we address in the final section of this paper.

Most important, we think prospective teachers must be placed in student teaching situations that provide the opportunity and support to teach in ways that match the NCTM's vision. They must be able to observe experienced teachers model appropriate teaching strategies, have time and incentives to prepare lessons that focus on conceptual knowledge, receive feedback on their lessons, and be protected from the accountability pressures that potentially restrict their opportunities. For this reason, student teaching placement decisions are crucial.

Ms. Daniels's experience in Ms. Santo's class provides some support for this recommendation. Ms. Santo did stress conceptual knowledge in her mathematics instruction. And although she typically emphasized abstract models in her own teaching, she provided feedback and suggestions about semiconcrete representations when Ms. Daniels chose (on her own) to include such representations in her teaching. Thus, her classroom provided a context in which Ms. Daniels could, and did, improve her teaching for conceptual knowledge. It is our speculation that this learning-to-teach context would have been further enhanced if Ms. Santo had modeled the use of concrete and semi-concrete representations, as well as abstract models, in teaching for conceptual knowledge.

Second, universities must figure out how to design the university component so that prospective teachers and their instructors have time to explore and develop approaches to teaching that are considered desirable. In the yearlong program we studied, there was not enough time and there were too many competing demands for new or innovative forms of mathematics teaching to receive much attention (Underhill, 1991). The student teachers spent 15 hours per week teaching in a placement school and 12 hours per week in university classes and seminars. And they had only one 12-week mathematics methods course to cover curriculum and instruction for K–8 mathematics. Further, the student teachers had limited opportunities in their placement sites to observe or participate in mathematics teaching and even fewer experiences with the type of mathematics instruction encouraged in the mathematics methods course.

One strategy to address issues raised in the preceding paragraphs is for university and public school personnel to work together more closely to develop a set of learning-to-teach opportunities for student teachers. For example, a methods course instructor might meet with a group of cooperating teachers to share their conceptions of teaching and learning to teach mathematics and to identify the kinds of experiences, support, and feedback to provide to student teachers. This type of collaborative effort is not easy. It requires a substantial initial investment of time and resources on the part of university and public school personnel, as well as a willingness to examine, seriously confront, and perhaps change fundamental beliefs and accepted practices about teaching and learning to teach mathematics. However, once in place, such collaborative relationships and shared commitments could provide more consistent opportunities for student teachers.

Even in programs developed collaboratively, however, competing priorities and tensions will exist. We think that mathematics methods instructors and cooperating teachers must take responsibility for helping student teachers recognize these tensions and explore ways to resolve them. For example, they might organize discussions of the factors that create pressures for teachers to teach for procedural knowledge. They might also devote time to brainstorming ways of creating and negotiating opportunities to try out teaching for conceptual knowledge in school contexts that are predominantly procedurally oriented.

Student teachers also are not without responsibilities to improve on the outcome we describe in this article. Although they are often assumed to be powerless, our data show that their collective response to instruction can subvert the opportunities provided by it. To some extent, student teachers can choose what they learn, or attempt to learn, from a particular course or

instructor. For student teachers to be receptive to new or innovative ideas proposed by their university instructors, they, too, must confront their previously developed beliefs and their existing knowledge about the nature of mathematics teaching, learning, and learning to teach. We reached a similar conclusion in our previous article (Borko et al., 1992). In that article, we offered specific suggestions for challenging the existing beliefs and knowledge base of prospective teachers and for developing their awareness of, and receptivity to, ideas about teaching and learning to teach that are compatible with current visions of mathematics education reform.

Mathematics education reformers believe in the power of teachers to effect change. "Teachers are key figures in changing the ways in which mathematics is taught and learned in schools" (NCTM, 1991, p. 2). However, our data suggest, at least in the case of novice teachers, that such changes are not likely to occur without more concerted and systematic efforts to organize contexts for learning to teach so that they consistently stress the priorities envisioned by the reform movement. We believe that our suggestions for modifying the university teacher education program and student teaching placements, and for creating greater compatibility between methods instruction and student teaching placements, represent such efforts. In particular, the collaboration can help to create contexts in which tensions are more likely to be recognized and resolved in ways that promote learning to teach for conceptual knowledge, and that reflect how procedural and conceptual knowledge are complexly intertwined, with both forms necessary for mathematical understanding.

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